The formation of Super soluble Groups

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Abstract: A class of finite groups is called saturated formation $\sigma$ when if $\frac{G}{\Phi(G)} \in \sigma$, then $G \in \sigma$ in which $\sigma$ is called a class of groups. The purpose of this paper is to show that the finite super soluble groups form a saturated formation. Also, using the $\sim G$ super solubility of some of the subgroups over the structure of finite groups, the Holder's theorem is generalized.

Keywords: Saturated formation, Super soluble group, $\sim G$ super solubility, Frattini subgroup

Introduction

Until now various studies have been performed over super soluble groups, we try to show in this paper that finite super soluble groups, form saturated formation. Holder showed that if each sylow subgroup of a finite group of $G$ is away, then $G$ is soluble. We attempt to generalize the Holder's theorem and illustrate that if each sylow subgroup of a finite group of $G$ is away, then $G$ is super soluble. It should be noted that all the groups are finite in this paper and the terminologies and symbols are common symbols.

Fundamental definitions and initial results

Definition 2-1: $\sigma$ is called a class of groups when group $G$ belongs to $\sigma$, then each isomorphic group with it belongs to $\sigma$ too. Also, when $G$ belongs to the $\sigma$ class, we call $G$ a $\sim \sigma$ group.

Definition 2-2: A class of finite groups is called formation $\sigma$ when each quotient group of a $\sim \sigma$ group is a $\sim \sigma$ group and if $\frac{G_{1}}{N_{1}}, \frac{G_{2}}{N_{2}}$ belongs to $\sigma$, and then $\frac{G}{N_{1} \cap N_{2}}$ belongs to $\sigma$.

Definition 2-3: A class of finite groups is called saturated formation $\sigma$ when if $\frac{G}{\Phi(G)} \in \sigma$, then $G \in \sigma$.

Definition 2-4: Assume that $G$ is a group in the sense $N \triangleleft G$ if $N$ has a normal series that all of its factors are away and each series expression of $N$ in $G$ is normal, then we call $N$ a super soluble $\sim G$.

Lemma 2-1: (1. 2. [2]). If $N \triangleleft G$, $N$, is a super soluble $\sim G$ and $\frac{G}{N}$ is super soluble, then $G$ is super soluble.

Lemma 2-2: (1. 3. [2]), Assume $N$ is a normal subgroup of the $G$ super soluble group, in this case $N$ is an expression of a $G$ super soluble series.

Lemma 2-3: (11 . 3 . 1 , [4]). Assume $G$ is a group away. In this case $G$ has a normal series like: $1 = G_{0} \leq G_{1} \leq \ldots \leq G_{n} = G$

Which each expression of that subgroup is a characterization of $G$ and for each $1 \leq i \leq n$, group $\frac{G_{i}}{G_{i-1}}$ is away from the first order.

Result 2-1: Assume that $G$ is a group and $N \triangleleft G$, if $N$ is away and $\frac{G}{N}$ is super soluble, then $G$ is super soluble.

Proof: According to lemma(2-3), $N$ has a normal series like:

$1 = N_{0} \leq N_{1} \leq \ldots \leq N_{n} = N$. Where each expression of that subgroup is a characterization of $N$ and for each $1 \leq i \leq n$, group $\frac{N_{i}}{N_{i-1}}$ is away from the first order and since each $N_{i}$ subgroup is a characterization of $N$ and $N \triangleleft G$, so $N_{i} \triangleleft G$. As a result, $N$ is super soluble $\sim G$. Now, according to Lemma (2-1), $G$ is super soluble.
Lemma 2 - 4: Assume G is a super soluble group and \( N \triangleleft G \), in this case N, is a super soluble \( \sim G \).
Proof: According to lemma (2-2), N is an expression of a super soluble series of G like: \( 1 = G_0 \leq G_1 \leq \ldots \leq G_k = N \leq G_k+1 \leq \ldots \leq G_n = G \) and since G is super soluble, each above series expression is normal in G and as a result each expression of the N series in G is normal and since all the G factors are away, so all the N factors are away, so N is a super soluble \( \sim G \).

Lemma 2-5: Assume G is a group and \( K \triangleleft N \triangleleft G \), if N is a super soluble \( \sim G \) then, K is super soluble \( \sim G \).
Proof: Since N is super soluble \( \sim G \) and \( K \triangleleft N \), so K in a super soluble \( \sim G \) series of N like : \( 1 = K_0 \leq K_1 \leq K_2 \leq \ldots \leq K_n \leq N \) where for each \( 1 \leq i \leq n + m \), group \( \frac{K_i}{K_{i-1}} \) is away from the first order and each \( K_i \triangleleft G \). Now it is clear that for each \( 1 \leq i \leq n \), group \( \frac{K_i}{K_{i-1}} \) is away from the first order and each \( K_i \triangleleft G \), so K is super soluble \( \sim G \).

Lemma 2 - 6: Assume G is a group and \( K \triangleleft N \triangleleft G \), if \( \frac{N}{K} \) is a super soluble \( \sim G \), then , N is super soluble \( \sim G \). In addition, if \( \frac{G}{K} \) is super soluble then, G is super soluble.
Proof: Since, \( \frac{N}{K} \) is super soluble \( \sim \frac{G}{K} \), so it has a super soluble series like:
\[
1 = \frac{N_0}{K} \leq \frac{N_1}{K} \leq \ldots \leq \frac{N_{k-1}}{K} \leq \frac{N}{K}
\]
where each expression of it is normal in \( \frac{G}{K} \) and each of its factor is away.
Now it is clear that each series expression of \( 1 = N_0 \leq N_1 \leq \ldots \leq N_n = N \) is normal in G and each of its factor is away, so N is super soluble \( \sim G \). According to Lemma (2-5), K is super soluble \( \sim G \). As a result according to Lemma (2-1), G is super soluble.

Lemma 2 - 7: Assume G is a group and \( K \triangleleft N \triangleleft G \), if N is super soluble \( \sim G \), then , \( \frac{N}{K} \) is super soluble \( \sim \frac{G}{K} \).

Lemma 2-8: Assume G is a group in the sense that a \( P \) has a normal and distant (away) sylow and \( \frac{G}{\Phi(P)} \) is super soluble, in this case G is super soluble.
Lemma2-9: Assume G is a group and \( N \triangleleft G \) in the sense that N is super soluble \( \sim G \), then , N is super soluble.

Theorem 2-1: Assume G is a group, if for each normal subgroup of N of G, \( \frac{G}{N} \) is super soluble, then, G is super soluble.

Theorem 2-2: Assume G is a group and \( N \triangleleft G \) in the sense \( \frac{G}{N} \) is super soluble and \( N \leq \Phi(G) \) then, G is super soluble.

Theorem 2-3: (11. 3 . 2 , [4]): Assume G is a super soluble group, in this case G has a normal series like : \( 1 = G_0 \leq G_1 \leq \ldots \leq G_n = G \) that for each \( 1 \leq i \leq n \), group \( \frac{G_i}{G_{i-1}} \) is away from the order of a prime number.

Result2-2 : Assume G is a super soluble group, if N is the normal minimal subgroup of G then N is away from the first order.

Theorem 2-4: (11 . 3 . 4 ,[4]): Assume G is a super soluble group, if \( p \) is the largest prime number that \( p \mid |G| \), then G has a normal super soluble sub group of the \( p \) order.

Result 2-3: Assume G is a super soluble group, if \( p \) is the largest prime number that \( p \mid |G| \), then , \( p^{-1} \) is the subgroup of the normal G sylow.

Lemma 2-7: Assume N is a normal soluble and non trivial subgroup of the G group. If each normal minimal subgroup of G that is in N is not in \( \Phi(G) \), then \( N \cap \Phi(G) = 1 \).

Proof: Assume \( N \cap \Phi(G) = K \) and \( K \neq 1 \), in this case \( K \leq \Phi(G) \) and since \( \Phi(G) \) is super soluble so, K is too super soluble . Now assume that \( p \) is the largest prime number that \( p \mid |K| \), in this case according to result (2-3), K has a
normal sylow subgroup of \(-p\) and as a result K has a normal minimal sylow subgroup of \(-p\) like \(P\). Also, \(-p\) are the normal sylow subgroups of the characterization, so the subgroup \(P\) is the characterization of K and because \(K \triangleleft G\), as a result \(P \triangleleft G\) that is in contradiction with the hypothesis. Because we found a normal minimal subgroup like \(P\) of G that is in \(\Phi(G)\), so \(N \cap \Phi(G) = 1\).

**Conclusion**

Lemma 3-1: Class of finite super soluble groups form saturated formation.

Proof: Assume it is not the case and G is a counterexample of the least order and \(\frac{G}{\Phi(G)} \in \sigma\). \((\frac{G}{\Phi(G)})\) is super soluble, in this case \(\Phi(G) \neq 1\) because if \(\Phi(G) = 1\) then the result is \(G \in \sigma\) which is a contradiction.

Now assume \(p\) to be the largest prime number that \(p \mid \frac{G}{\Phi(G)}\), in this case according to theorem (2-4), \(\frac{G}{\Phi(G)}\) has a normal super soluble group like \(\frac{N}{\Phi(G)}\) and as a result \(N\) is a normal soluble subgroup of G. Now we face two stages:

Stage 1: If any normal minimal subgroup of G that is in N, is not in \(\Phi(G)\), then according to Lam (2-7), \(N \cap \Phi(G) = 1\), and also, according to that fact that \(\frac{N}{\Phi(G)} \leq \frac{G}{\Phi(G)}\), so \(\frac{N}{\Phi(G)}\) is super soluble and as a result

\[
\frac{G}{\Phi(G)} / \frac{N}{\Phi(G)}
\]

is too super soluble and according to the second isomorphism theorem \(\frac{G}{\Phi(G)} / \frac{N}{\Phi(G)} \cong \frac{G}{N}\). Therefore, \(\frac{G}{N}\) is super soluble and because \(\frac{G}{\Phi(G)}\) is super soluble too, hence, \(\frac{G}{N} \times \frac{G}{\Phi(G)}\) is super soluble and since

\[
\frac{G}{N \cap \Phi(G)} \leq \frac{G}{N} \times \frac{G}{\Phi(G)}
\]

we have \(\frac{G}{N \cap \Phi(G)}\) that is super soluble. As a result \(G\) is too super soluble which is a contradiction.

Stage 2: There is otherwise a normal minimal subgroup of G like K that \(K \leq N \cap \Phi(G)\). Now, assuming \(N \cap \Phi(G) = K\), we have \(\frac{G}{K} \leq \frac{G}{N} \times \frac{G}{\Phi(G)}\) and as a result \(\frac{G}{K}\) is super soluble and because \(K \leq \Phi(G)\), according to theorem (2-2), G is super soluble which is a contradiction too.

These contradictions indicate that such a counter example does not exist and as a result G is super soluble. Hence, the class of finite super soluble groups forms saturated formation.

Lemma 3-1: Assume that G is a group and \(N \triangleleft \Phi(G)\) and \(\Phi(G)\), G is super soluble. If \(\frac{G}{N}\) is super soluble, then G is super soluble.

Proof: According to Lemma (2-5), N is a super soluble \(-G\). Now according to Lemma(2-1), G is super soluble.

Theorem 2-3: Assume \(P\) is a \(-p\) sylow subgroup of the finite group G in which \(p\) is the smallest prime number that makes \(|G|\) be a factor. In this case if \(P\) is away, \(P\) has a normal complement in G.

Theorem 1-3: If each sylow subgroup of the finite group G is away, then G is super soluble.

Proof: Proof is with induction in relation to \(|G|\). In case \(|G| = 1\), mandate is clear. Assume \(|G| > 1\) and \(p\) is the smallest prime number that \(p \mid |G|\). Now, we assume \(P\) to be a \(-p\) sylow subgroup of G. According to theorem (2-3), G has a normal non trivial subgroup like K in the sense, \(\frac{G}{K} \cong P\). Now, it is clear that \(\frac{G}{K}\) is super soluble and since each \(-p\) sylow subgroup of K is in a sylow subgroup of G and the subgroups of the away groups are away, So, \(-p\) sylow subgroups of K are away too. Because, \(|K| \sim |G|\), then as per the induction hypothesis K is super soluble. Now, we
illustrate that $K$ is super soluble $\sim G$. Assume that $N$ is a $P$ maximal subgroup of $K$, hence, $N$ is away and $N$ subgroup is a characterization of $K$, so $N \triangleleft G$ and $|K : N| = p$, so $K$ has a super soluble $\sim G$ series like $\cdots$ an hence, $1 \leq N \leq K$ $K$ is super soluble $\sim G$. Now according to Lemma (1-2), $G$ is super soluble.

References
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